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Valuations on convex sets, non-commutative determinants, and pluripotential theory

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Abstract

A new method of constructing translation invariant continuous valuations on convex subsets of the quaternionic space $\mathbb{H}^n$ is presented. In particular new examples of $Sp(n)Sp(1)$-invariant translation invariant continuous valuations are constructed. This method is based on the theory of plurisubharmonic functions of quaternionic variables developed by the author in two previous papers (Bull. Sci. Math. 127 (1) (2003) 1; J. Geom. Anal. 13 (2) (2003) 183).

Keywords: Convex sets; Valuations; Plurisubharmonic functions; Non-commutative determinants; Quaternions

0. Introduction

The goal of this paper is to present a new method of constructing translation invariant continuous valuations on convex subsets of the quaternionic space $\mathbb{H}^n$. As an application of this method we obtain new examples of $Sp(n)Sp(1)$-invariant translation invariant continuous valuations. This method is based on the theory of plurisubharmonic functions of quaternionic variables developed by the author in two previous papers [5,6]. The main results of the paper are Theorem 4.2.1 and its immediate Corollary 4.2.2.

Let us remind basic notions of the theory of valuations on convex sets referring for more details to the surveys by McMullen [30] and McMullen and Schneider [31]. Let V be a finite-dimensional real vector space. Let $\mathcal{K}(V)$ denote the class of all convex compact subsets of V.

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Definition 0.1.1. (a) A function $\phi : \mathcal{K}(V) \to \mathbb{C}$ is called a valuation if for any $K_1, K_2 \in \mathcal{K}(V)$ such that their union is also convex one has

$$\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).$$

(b) A valuation ϕ is called continuous if it is continuous with respect the Hausdorff metric on $\mathcal{K}(V)$.

Remind that the Hausdorff metric d_{H} on $\mathcal{K}(V)$ depends on a choice of a Euclidean metric on V and it is defined as follows: $d_{\mathrm{H}}(A, B) := \inf\{\varepsilon > 0 | A \subset (B)_\varepsilon \text{ and } B \subset (A)_\varepsilon\}$, where $(U)_\varepsilon$ denotes the ε-neighborhood of a set U. Then $\mathcal{K}(V)$ becomes a locally compact space, and the topology on $\mathcal{K}(V)$ induced by the Hausdorff metric does not depend on the choice of the Euclidean metric on V.

The theory of valuations has numerous applications in convexity and integral geometry (see e.g. [4,7,21,28,34]). Classification results on valuations invariant under specific groups are of particular importance for applications. The famous result of Hadwiger [21] describes explicitly isometry invariant continuous valuations on a Euclidean space $\mathbb{R}^n$. For discussion of applications of Hadwiger's result to integral geometry in Euclidean spaces we refer to the books Hadwiger [21], Klain and Rota [28], and the survey article by Hug and Schneider [25].

It was shown by the author in [3] that if G is a compact subgroup of the orthogonal group $O(n)$ acting transitively on the unit sphere then the space of G-invariant translation invariant continuous valuations (let us denote this space by Val^G) is finite dimensional. Hence in this case one may hope to obtain an explicit description of the space Val^G. It was shown in [8,9] (see also [7]) that for a group G satisfying the above assumptions the space Val^G has various remarkable properties: it has a canonical structure of a commutative associative graded algebra satisfying the Poincaré duality, and if $-Id \in G$ it satisfies the hard Lefschetz-type theorem. Remind also that there is an explicit classification of compact connected groups acting transitively and effectively on the sphere [13,14,32]. Namely there are 6 infinite series $SO(n), U(n), SU(n), Sp(n), Sp(n)Sp(1), Sp(n)U(1)$, and 3 exceptions $G_2, Spin(7), Spin(9)$.

The case of Val^G for the groups $G = O(n)$ and $G = SO(n)$ was completely classified by Hadwiger [21]. In [7] the author has obtained an explicit classification of translation invariant $U(n)$-invariant continuous valuations on an Hermitian space $\mathbb{C}^n$; in that article also some applications to integral geometry in Hermitian spaces were obtained. The space of $SU(2)$-invariant valuations on $\mathbb{C}^2$ was described in [10].

In this article we study the case of translation invariant $Sp(n)Sp(1)$-invariant valuations on the quaternionic space $\mathbb{H}^n$. Though we do not obtain a complete classification, we construct new non-trivial examples of such valuations using a new method. This method is based on the theory of plurisubharmonic functions of quaternionic variables developed by the author in [5,6]. Motivated by applications to valuations, we obtain in this paper further results in this theory. In order to explain our main results let us first discuss their complex analogs which are more classical and well known.

Let $\mathbb{C}^n$ be a Hermitian space with the Hermitian product $(\cdot,\cdot)$. For a convex compact subset $K \in \mathcal{K}(\mathbb{C}^n)$ let h_K denote its supporting functional (remind that $h_K : \mathbb{C}^n \to \mathbb{R}$, and $h_K(x) := \sup_{y\in K}(x, y)$). Let us fix an integer $1 \leqslant l \leqslant n$. Let us fix a continuous compactly supported $(n-l, n-l)$-form ψ on $\mathbb{C}^n$. Define

$$\phi(K) := \int_{\mathbb{C}^n} (dd^c h_K)^l \wedge \psi. \tag{1}$$

Theorem 4.1.3 of this paper claims that ϕ is a translation invariant continuous valuation. Note that since the function h_K is not necessarily smooth (but it is convex, and hence continuous plurisubharmonic) the expression $(dd^c h_K)^l$ should be understood in the sense of currents using the Chern–Levine–Nirenberg theorem [15]. The continuity property of ϕ is not obvious and also follows from the same result [15]. The property of valuation is not evident; it is a consequence of a more general result about plurisubharmonic functions due to Blocki [12] (this was kindly explained to us by N. Levenberg).

The main result of this article is a construction of a quaternionic analogue on $\mathbb{H}^n$ of valuations of form (1) (Theorem 4.2.1). This construction is based on the theory of plurisubharmonic (psh) functions of quaternionic variables. The notion of psh function of quaternionic variables was introduced by the author in [5] and independently by Henkin [23]. This class of functions was studied further by the author in [6] where the quaternionic Monge–Ampère equations were introduced and investigated. A quaternionic version of the Chern–Levine–Nirenberg theorem was proved in [5]. In this paper we prove a refined version of that result (Theorem 3.1.6). (The refinement is approximately as follows: our previous result from [5] corresponds in the complex case to establishing of some properties of the current $(dd^c h)^n$ on $\mathbb{C}^n$ where h is a continuous complex psh function, and Theorem 3.1.6 of this paper corresponds to the current $(dd^c h)^l$ with $0 \leqslant l \leqslant n$.) This required to introduce a quaternionic analogue of the notion of positive current (Section 3.1). Then we prove a quaternionic analogue of Blocki's formula (Theorem 3.2.1).

Note that most of the results of this article and [5,6] as well make use of the notion of non-commutative determinant, particularly the Moore determinant of quaternionic hyperhermitian matrices which is reviewed in Section 1.2. For the purposes of this article it was necessary to present a new, coordinate free, construction of the Moore determinant. This is done in Section 2.1. Moreover, we have constructed a quaternionic analogue of the algebra of exterior forms of type (p, p) on a complex space (Definition 2.1.12). Note also that it was shown in [18] that the Moore determinant can be expressed via Gelfand–Retakh quasideterminants first introduced in [17] and which generalize most of the known notions of non-commutative determinants (see also Remark 1.2.18 in Section 1.2 of this paper). For the details we refer to the recent survey [19], and for more details on the quaternionic case we refer to [18].

Let us describe briefly the content of the paper. Section 1 does not contain new results. In Section 1.1 we review some relevant facts from the complex analysis. In Section 1.2 we remind the necessary definitions and facts about hyperhermitian matrices and the Moore determinant. In Section 1.3 we summarize the relevant definitions and results about psh functions of quaternionic variables following [5].

Section 2 contains some new constructions from quaternionic linear algebra. Section 2.1 is purely algebraic. There we describe a coordinate-free construction of the Moore determinant. Next for a right quaternionic $\mathbb{H}$-module V of finite dimension we construct a graded algebra $\Omega^\bullet(V)$ analogous to the graded algebra of exterior forms of type (p, p) on a complex space. In Section 2.2 we introduce the notions of weakly and strongly positive elements in $\Omega^\bullet(V)$.

In Section 3 we study further psh functions of quaternionic variables. In Section 3.1 we prove a quaternionic analogue of the Chern–Levine–Nirenberg theorem refining our previous result from [5]. This theorem concerns positive currents with values in $\Omega^\bullet(V)$. In Section 3.2 we obtain a quaternionic analogue of the Blocki formula.

Section 4 contains applications of the above results to the theory of valuations. In Section 4.1 we remind Kazarnovskii's pseudovolume and its generalizations following [26,27]. In Section 4.2 we obtain quaternionic analogues of these valuations, i.e. the main results of this paper (Theorem 4.2.1 and Corollary 4.2.2).

1. Background

In Section 1.1, we remind some definitions and facts from complex analysis which are very classical. In Section 1.2, we remind some facts from the theory of non-commutative determinants, the exposition will follows essentially [5]. In Section 1.3, we review the theory of plurisubharmonic functions of quaternionic variables developed by the author in [5] and which is based on the theory of non-commutative determinants. This section does not contain new results.

1.1. Some complex analysis

Let us remind the definition of a plurisubharmonic function of complex variables (see e.g. Lelong's book [29] for more details). Let Ω be an open subset in $\mathbb{C}^n$.

Definition 1.1.1. A real-valued function $u : \Omega \to \mathbb{R}$ is called psh if it is upper semi-continuous and its restriction to any *complex* line is subharmonic.

Recall that upper semi-continuity means that $u(x_0) \geqslant \limsup_{x \to x_0} u(x)$ for any $x_0 \in \Omega$. We will denote by $P(\Omega)$ the class of psh functions in the domain Ω, and the class of continuous functions in Ω will be denoted by $C(\Omega)$.

Remark 1.1.2. If in the above definition one replaces the word "complex" by the word "real" everywhere then one obtains a definition equivalent to the usual definition of *convex* function.

The following result is due to Chern et al. [15] (its real analogue is due to Aleksandrov [2]).

Theorem 1.1.3 (*Chern et al. [15]*). *Let $\Omega \subset \mathbb{C}^n$ be an open subset. For any function $u \in C(\Omega) \cap P(\Omega)$ one can define a non-negative measure denoted by* $\det(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j})$ *which is uniquely characterized by the following two properties*:

(a) *if $u \in C^2(\Omega)$ then it has the obvious meaning*;
(b) *if $u_N \to u$ uniformly on compact subsets in Ω, and u_N, $u \in C(\Omega) \cap P(\Omega)$, then*

$$\det\left(\frac{\partial^2 u_N}{\partial z_i \partial \bar{z}_j}\right) \xrightarrow{w} \det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right),$$

where the convergence of measures in understood in the sense of weak convergence of measures.

Remark 1.1.4. In fact it is easy to see from the definition that the uniform limit of functions from $P(\Omega) \cap C(\Omega)$ also belongs to this class.

1.2. Non-commutative determinants

For our purposes we will need the notion of Moore determinant. The exposition follows essentially [5]. For more developed theory of non-commutative determinants, so-called quasideterminants of Gelfand–Retakh, we refer to the survey [19]. In [18] there was established the connection of quasideterminants to the Moore determinant. A good survey of quaternionic determinants is [11].

Definition 1.2.1. A *hyperhermitian semilinear form* on V is a map $a : V \times V \to \mathbb{H}$ satisfying the following properties:

(a) a is additive with respect to each argument;
(b) $a(x, y \cdot q) = a(x, y) \cdot q$ for any $x, y \in V$ and any $q \in \mathbb{H}$;
(c) $a(x, y) = \overline{a(y, x)}$.

Example 1.2.2. Let $V = \mathbb{H}^n$ be the standard coordinate space considered as right vector space over $\mathbb{H}$. Fix a *hyperhermitian* $n \times n$-matrix $(a_{ij})_{i,j=1}^n$, i.e. $a_{ij} = \bar{a}_{ji}$, where $\bar{x}$ denotes the usual quaternionic conjugation of x. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define

$$A(x, y) = \sum_{i,j} \bar{x}_i a_{ij} y_j$$

(notice the order of the terms!). Then A is a hyperhermitian semilinear form on V.

The set of all hyperhermitian $n \times n$-matrices will be denoted by $\mathcal{H}_n$. Then $\mathcal{H}_n$ a vector space over $\mathbb{R}$.

In general one has the following standard claims.

Claim 1.2.3. *Fix a basis in a finite-dimensional right quaternionic vector space V. Then there is a natural bijection between the space of hyperhermitian semilinear forms on V and the space* $\mathcal{H}_n$ *of* $n \times n$*-hyperhermitian matrices.*

This bijection is in fact described in previous Example 1.2.2.

Claim 1.2.4. *Let A be the matrix of the given hyperhermitian form in the given basis. Let C be transition matrix from this basis to another one. Then the matrix* A' *of the given form in the new basis is equal*

$$A' = C^* AC.$$

Remark 1.2.5. Note that for any hyperhermitian matrix A and for any matrix C the matrix C^*AC is also hyperhermitian. In particular the matrix C^*C is always hyperhermitian.

Definition 1.2.6. A hyperhermitian semilinear form a is called *positive definite* if $a(x,x) > 0$ for any non-zero vector x. Similarly a is called *non-negative definite* if $a(x,x) \geqslant 0$ for any vector x.

Let us fix on our quaternionic right vector space V a positive definite hyperhermitian form $(\cdot,\cdot)$. The space with fixed such a form will be called *hyperhermitian* space.

For any quaternionic linear operator $\phi\colon V \to V$ in hyperhermitian space one can define the adjoint operator $\phi^*\colon V \to V$ in the usual way, i.e. $(\phi x, y) = (x, \phi^* y)$ for any $x, y \in V$. Then if one fixes an orthonormal basis in the space V then the operator ϕ is selfadjoint if and only if its matrix in this basis is hyperhermitian.

Claim 1.2.7. *For any selfadjoint operator in a hyperhermitian space there exists an orthonormal basis such that its matrix in this basis is diagonal and real.*

Now we are going to remind the definition of the Moore determinant of hyperhermitian matrices. The definition below is different from the original one [33] but equivalent to it.

Any quaternionic matrix $A \in M_n(\mathbb{H})$ can be considered as a matrix of an $\mathbb{H}$-linear endomorphism of $\mathbb{H}^n$. Identifying $\mathbb{H}^n$ with $\mathbb{R}^{4n}$ in the standard way we get an $\mathbb{R}$-linear endomorphism of $\mathbb{R}^{4n}$. Its matrix in the standard basis will be denoted by ${}^{\mathbb{R}}A$, and it is called the *realization* of A. Thus ${}^{\mathbb{R}}A \in M_{4n}(\mathbb{R})$.

Let us consider the entries of A as formal variables (each quaternionic entry corresponds to four commuting real variables). Then $\det({}^{\mathbb{R}}A)$ is a homogeneous polynomial of degree $4n$ in $n(2n-1)$ real variables. Let us denote by Id the identity matrix. One has the following result.

Theorem 1.2.8. *There exists a polynomial P defined on the space* $\mathcal{H}_n$ *of all hyperhermitian* $n \times n$*-matrices such that for any hyperhermitian* $n \times n$*-matrix A one has* $\det({}^{\mathbb{R}}A) = P^4(A)$ *and* $P(Id) = 1$*. P is defined uniquely by these two properties. Furthermore P is homogeneous of degree n and has integer coefficients.*

Thus for any hyperhermitian matrix A the value $P(A)$ is a real number, and it is called the *Moore determinant* of the matrix A. The explicit formula for the Moore determinant was given by Moore [33] (see also [11]). From now on the Moore determinant of a matrix A will be denoted by $\det A$. This notation should not cause any confusion with the usual determinant of real or complex matrices due to part (i) of the next theorem.

Theorem 1.2.9. (i) *The Moore determinant of any complex hermitian matrix considered as quaternionic hyperhermitian matrix is equal to its usual determinant.*

(ii) *For any hyperhermitian $n \times n$-matrix A and any matrix $C \in M_n(\mathbb{H})$*

$$\det(C^* A C) = \det A \cdot \det(C^* C).$$

Example 1.2.10. (a) Let $A = diag(\lambda_1, \dots, \lambda_n)$ be a diagonal matrix with real λ_i's. Then A is hyperhermitian and the Moore determinant $\det A = \prod_{i=1}^n \lambda_i$.

(b) A general hyperhermitian 2×2-matrix A has the form

$$A = \begin{bmatrix} a & q \\ \bar{q} & b \end{bmatrix},$$

where $a, b \in \mathbb{R}$, $q \in \mathbb{H}$. Then $\det A = ab - q\bar{q}$.

Definition 1.2.11. A hyperhermitian $n \times n$-matrix $A = (a_{ij})$ is called *positive* (resp. *non-negative*) *definite* if for any non-zero vector $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$ one has $\xi^* A \xi = \sum_{ij} \bar{\xi}_i a_{ij} \xi_j > 0$ (resp. $\geqslant 0$).

Claim 1.2.12. *Let A be a non-negative* (*resp. positive*) *definite hyperhermitian matrix. Then* $\det A \geqslant 0$ ($resp.$ $\det A > 0$).

Let us remind now the definition of the mixed discriminant of hyperhermitian matrices in analogy with the case of real symmetric matrices [1].

Definition 1.2.13. Let $A_1, \dots, A_n$ be hyperhermitian $n \times n$-matrices. Consider the homogeneous polynomial in real variables $\lambda_1, \dots, \lambda_n$ of degree n equal to $\det(\lambda_1 A_1 + \dots + \lambda_n A_n)$. The coefficient of the monomial $\lambda_1 \cdots \lambda_n$ divided by $n!$ is called the *mixed discriminant* of the matrices $A_1, \dots, A_n$, and it is denoted by $\det(A_1, \dots, A_n)$.

Note that the mixed discriminant is symmetric with respect to all variables, and linear with respect to each of them, i.e.

$$\det(\lambda A_1' + \mu A_1'', A_2, \dots, A_n) = \lambda \cdot \det(A_1', A_2, \dots, A_n) + \mu \cdot \det(A_1'', A_2, \dots, A_n)$$

for any *real* λ, μ. Note also that $\det(A, \dots, A) = \det A$.

Theorem 1.2.14. *The mixed discriminant of positive* (*resp. non-negative*) *definite matrices is positive* (*resp. non-negative*).

We will need also the following proposition which is essentially well known but we do not have an exact reference.

Proposition 1.2.15. *Let A be a quaternionic $n \times n$-matrix. Then*

$$\det({}^{\mathbb{R}}A) = (\det(A^*A))^2 = (\det(AA^*))^2,$$

where the first determinant denotes the usual determinant of the real $(4n \times 4n)$-matrix ${}^{\mathbb{R}}A$, and the second and the third determinants denote the Moore determinant.

Proof. We may assume that A is invertible, i.e. $A \in GL_n(\mathbb{H})$ (otherwise both sides are equal to 0). The maximal compact subgroup of $GL_n(\mathbb{H})$ is $Sp(n)$. There exists a decomposition $A = U \cdot D \cdot V$ where $U, V \in Sp(n)$ and D is a real diagonal matrix, $D = diag(\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \mathbb{R}$. Since $Sp(n) \subset SO(4n)$ we have

$$\det({}^{\mathbb{R}}A) = \det({}^{\mathbb{R}}D) = (\prod_{i=1}^{n} \lambda_i)^4.$$

On the other hand

$$\det(AA^*) = \det(UD^2U^*) = (\prod_{i=1}^{n} \lambda_i)^2,$$

where the last equality follows from Theorem 1.2.9 (ii) and Example 1.2.10. Hence $\det({}^{\mathbb{R}}A) = (\det(AA^*))^2$. Similarly, one shows that $\det({}^{\mathbb{R}}A) = (\det(A^*A))^2$. $\square$

Let us introduce more notation. Let A be any hyperhermitian $n \times n$-matrix. For any subset $I \subset \{1, \dots, n\}$ the minor $M_I(A)$ of A which is obtained by deleting the rows and columns with indexes from the set I, is clearly hyperhermitian. (For instance $M_\emptyset(A) = A$.) For $I = \{1, \dots, n\}$ set $\det M_{\{1,\dots,n\}} := 1$. We will need a lemma.

Lemma 1.2.16. *Let $T := \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix}$ be a real diagonal $n \times n$-matrix. Let $A_1, \dots, A_{n-1}$ be hyperhermitian $n \times n$-matrices. Then*

$$\det(T, A_1, \dots, A_{n-1}) = \frac{1}{n} \sum_{i=1}^{n} t_i \det(M_{\{i\}}(A_1), \dots, M_{\{i\}}(A_{n-1})).$$

Proof. In [5, Proposition 1.1.11], it was shown that for a matrix T as above and for any hyperhermitian $n \times n$-matrix A one has

$$\det(A+T) = \sum_{I \subset \{1,\dots,n\}} \left(\prod_{i \in I} t_i\right) \cdot \det M_I(A). \tag{2}$$

Lemma 1.2.16 follows from this formula and the definition of the mixed discriminant. □

The following lemma also will be used later.

Lemma 1.2.17. *Let $A_1, \dots, A_n \geqslant 0$ be non-negative hyperhermitian $n \times n$-matrices. Then*

$$\det(A_1, \dots, A_n) \leqslant \det\left(\sum_{i=1}^n A_i\right).$$

Proof. First observe that Theorem 1.2.14 implies that if $B_1, \dots, B_n, C_1, \dots, C_n$ are hyperhermitian $(n \times n)$-matrices and

$$B_i \leqslant C_i \text{ for } i = 1, \dots, n$$

then

$$\det(B_1, \dots, B_n) \leqslant \det(C_1, \dots, C_n).$$

Now let us take $B_i := A_i$ and $C_1 = \cdots = C_n := \sum_{i=1}^n A_i$. Lemma 1.2.17 is proved. □

Remark 1.2.18. The Moore determinant of hyperhermitian quaternionic matrices behaves exactly as the usual determinant of complex hermitian or real symmetric matrices from all points of view. There is also a notion of the Dieudonné determinant [16] of an arbitrary quaternionic matrix which behaves exactly like the absolute value of the usual determinant of complex or real matrices. It was applied to the theory of quaternionic psh functions in author's paper [5]. Implicitly the Dieudonné determinant is also used in this article since the proof in [5] of Theorem 1.3.3 was based on it. The Dieudonné determinant can also be expressed via Gelfand–Retakh quasideterminants (see [19]).

1.3. Quaternionic pluripotential theory

The notion of psh function of quaternionic variables was introduced by the author in [5] and independently by Henkin [23]. The exposition here follows [5] where the definition of psh function of quaternionic variables is presented in the form suggested by Henkin, and parallel to the complex case. Let Ω be a domain in $\mathbb{H}^n$.

Definition 1.3.1. A real-valued function $u: \Omega \to \mathbb{R}$ is called quaternionic psh if it is upper semi-continuous and its restriction to any right *quaternionic* line is subharmonic.

The class of psh functions in Ω will be denoted by $P(\Omega)$, and the class of continuous functions will be denoted by $C(\Omega)$.

Let q be a quaternionic coordinate,

$$q = t + ix + jy + kz,$$

where t, x, y, z are real numbers. Consider the following operators defined on the class of smooth $\mathbb{H}$-valued functions of the variable $q \in \mathbb{H}$:

$$\frac{\partial}{\partial \bar{q}} f := \frac{\partial f}{\partial t} + i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}, \text{ and}$$

$$\frac{\partial}{\partial q} f := \overline{\frac{\partial}{\partial \bar{q}} \bar{f}} = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} i - \frac{\partial f}{\partial y} j - \frac{\partial f}{\partial z} k.$$

Note that $\frac{\partial}{\partial \bar{q}}$ is called sometimes Cauchy–Riemann–Moisil–Fueter operator, or sometimes Dirac–Weyl operator, or just Dirac operator. It is easy to see that $\frac{\partial}{\partial \bar{q}}$ and $\frac{\partial}{\partial q}$ commute, and if f is a *real-valued* function then

$$\frac{\partial}{\partial \bar{q}} \frac{\partial}{\partial q} f = \Delta f = \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f.$$

For any real-valued C^2-smooth function f the matrix $(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j})_{i,j=1}^n$ is obviously hyperhermitian. Note also that the operators $\frac{\partial}{\partial q_i}$ and $\frac{\partial}{\partial \bar{q}_j}$ commute. One can easily check the following identities.

Proposition 1.3.2. *Let $f: \mathbb{H}^n \to \mathbb{H}$ be a smooth function. Then for any $\mathbb{H}$-linear transformation A of $\mathbb{H}^n$ (as right $\mathbb{H}$-vector space) one has the identities*

$$\left(\frac{\partial^2 f(Aq)}{\partial \bar{q}_i \partial q_j}\right) = A^* \left(\frac{\partial^2 f}{\partial \bar{q}_i \partial q_j}(Aq)\right) A.$$

We have the following quaternionic analogue of Theorem 1.1.3.

Theorem 1.3.3 (*Alesker [5]*). *For any function $u \in C(\Omega) \cap P(\Omega)$ one can define a non-negative measure denoted by* $\det(\frac{\partial^2 u}{\partial q_i \partial \bar{q}_j})$ *which is uniquely characterized by the* *following two properties*:

(a) *if* $u \in C^2(\Omega)$ *then it has the obvious meaning*;
(b) *if* $u_N \to u$ *uniformly on compact subsets in* Ω, *and* $u_N,\ u \in C(\Omega) \cap P(\Omega)$, *then*

$$\det\left(\frac{\partial^2 u_N}{\partial q_i \partial \bar{q}_j}\right) \xrightarrow{w} \det\left(\frac{\partial^2 u}{\partial q_i \partial \bar{q}_j}\right),$$

where the convergence of measures is understood in the sense of weak convergence of measures.

Remark 1.3.4. In fact it is easy to see from the definition that the uniform limit of functions from $P(\Omega) \cap C(\Omega)$ also belongs to this class.

2. More quaternionic linear algebra

2.1. The space of forms

The goal of this subsection is to construct a quaternionic analogue of the spaces of complex forms of type (k, k). We will also present another (coordinate free) construction of the Moore determinant.

Let W be a right finite-dimensional $\mathbb{H}$-module, $n = \dim_{\mathbb{H}} W$. Let $\bar{W}$ denote the quaternionic conjugate space of W. Recall that $\bar{W}$ is a left $\mathbb{H}$-module, it coincides with W as a group, and the multiplication by scalars from $\mathbb{H}$ is given by $q \cdot w := w \cdot \bar{q}$. Consider the tensor product $W \otimes_{\mathbb{H}} \bar{W}$. On this space one has an involution $\sigma : W \otimes_{\mathbb{H}} \bar{W} \to W \otimes_{\mathbb{H}} \bar{W}$ defined by $\sigma(x \otimes y) = y \otimes x$.

Theorem 2.1.1. *Let* W *be a right* $\mathbb{H}$*-module*, $\dim_{\mathbb{H}} W = n$. *Then the space* $Sym^n((W \otimes_{\mathbb{H}} \bar{W})^\sigma)$ *has unique* $Aut_{\mathbb{H}} W$*-invariant subspace of* (*real*) *codimension* 1.

We postpone the proof of this theorem.

Definition 2.1.2. The one-dimensional quotient space of $Sym^n(W \otimes_{\mathbb{H}} \bar{W})$ by the subspace from Theorem 2.1.1 will be denoted by $M(W)$. The quotient map $M : Sym^n((W \otimes_{\mathbb{H}} \bar{W})^\sigma) \to M(W)$ will be called the *Moore map*.

Let us explain why we call the map M the Moore map. In fact it coincides with the Moore determinant in the following sense. Assume we are given on W a hyperhermitian positive definite form. Then it identifies the dual space $W^* := Hom_{\mathbb{R}}(W, \mathbb{R}) = Hom_{\mathbb{H}}(W, \mathbb{H})$ with $\bar{W}$. Then $W \otimes_{\mathbb{H}} \bar{W} \simeq W \otimes_{\mathbb{H}} W^* = End_{\mathbb{H}}(W, W)$. Under this identification $(W \otimes_{\mathbb{H}} \bar{W})^\sigma$ corresponds to selfadjoint endomorphisms of W. For any selfadjoint operator A, $M(A, \ldots, A) \in M(W)$. The space $M(W)$ can be identified with $\mathbb{R}$ as $Sp(n)$-module (since the last group is compact and connected, and hence does not have non-trivial real-valued characters) when $1 \in \mathbb{R}$ corresponds to $M(Id, \ldots, Id)$. Under this identifications $M(A, \ldots, A)$ is equal to the Moore determinant of A.

Before we prove Theorem 2.1.1 we need some preparations. Let us denote by ${}^{\mathbf{C}}\mathbb{H}$ the algebra of complex quaternions: ${}^{\mathbf{C}}\mathbb{H} := \mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}$. Recall that there exists an isomorphism of algebras ${}^{\mathbf{C}}\mathbb{H} \simeq M_2(\mathbb{C})$. Denote also ${}^{\mathbf{C}}W := W \otimes_{\mathbb{R}} \mathbb{C}$, and ${}^{\mathbf{C}}\bar{W} := \bar{W} \otimes_{\mathbb{R}} \mathbb{C}$. Let us fix a simple non-trivial right ${}^{\mathbf{C}}\mathbb{H}$-module T (which is unique up to non-canonical isomorphism since ${}^{\mathbf{C}}\mathbb{H}$ is a central simple algebra). Then $\dim_{\mathbb{C}} T = 2$.

Remark 2.1.3. Any finitely generated ${}^{\mathbf{C}}\mathbb{H}$-module is isomorphic to a direct sum of finitely many copies of T. This is a general fact for modules over central simple algebras (see e.g. [35, Chapter IX, Section 1, Proposition 1]).

Lemma 2.1.4. *There exists an isomorphism of* $\mathbb{C}$*-vector spaces*

$$(W \otimes_{\mathbb{H}} \bar{W})^{\sigma} \otimes_{\mathbb{R}} \mathbb{C} = \wedge^2(Hom_{{}^{\mathbf{C}}\mathbb{H}}(T, {}^{\mathbf{C}}W))$$

which commutes with the action of $Aut_{\mathbb{H}} W$.

Remark 2.1.5. (1) This isomorphism commutes with the natural action of $(Aut_{\mathbb{H}} W)\otimes_{\mathbb{R}} \mathbb{C} \simeq GL_{2n}(\mathbb{C})$.

(2) $\dim_{\mathbb{C}} Hom_{{}^{\mathbf{C}}\mathbb{H}}(T, {}^{\mathbf{C}}W) = 2n$. Indeed by Remark 2.1.3 the ${}^{\mathbf{C}}\mathbb{H}$-module ${}^{\mathbf{C}}W$ is isomorphic to a direct sum of $2n$ copies of the module T. Also one has isomorphism of algebras $End_{{}^{\mathbf{C}}\mathbb{H}}(T, T) = \mathbb{C}$.

Proof of Lemma 2.1.4. Note that the vector space in the left-hand side is canonically isomorphic to $({}^{\mathbf{C}}W \otimes_{{}^{\mathbf{C}}\mathbb{H}} {}^{\mathbf{C}}\bar{W})^{\sigma}$. Denote $Z := Hom_{{}^{\mathbf{C}}\mathbb{H}}(T, {}^{\mathbf{C}}W)$. Then the evaluation map $T \otimes_{\mathbb{C}} Z \to {}^{\mathbf{C}}W$ is an isomorphism. Then ${}^{\mathbf{C}}\bar{W}$ can be identified with $\bar{T} \otimes_{\mathbb{C}} Z$. Then ${}^{\mathbf{C}}W \otimes_{{}^{\mathbf{C}}\mathbb{H}} {}^{\mathbf{C}}\bar{W} = Z \otimes_{\mathbb{C}} (T \otimes_{{}^{\mathbf{C}}\mathbb{H}} \bar{T}) \otimes_{\mathbb{C}} Z$. But $T \otimes_{{}^{\mathbf{C}}\mathbb{H}} \bar{T}$ is one dimensional. Hence ${}^{\mathbf{C}}W \otimes_{{}^{\mathbf{C}}\mathbb{H}} {}^{\mathbf{C}}\bar{W}$ is isomorphic to $Z \otimes_{\mathbb{C}} Z$ and this identification commutes with the action of $(Aut_{\mathbb{H}} W) \otimes_{\mathbb{R}} \mathbb{C} = Aut_{\mathbb{C}} Z$. But $Z \otimes_{\mathbb{C}} Z$ decomposes uniquely under the action of $GL(Z)$ into two irreducible subspaces:

$$Z \otimes Z = \wedge^2 Z \oplus Sym^2 Z.$$

Hence on $Z \otimes Z$ there are only two non-trivial involutions commuting with the action of $Aut_{\mathbb{C}} Z$. It is easy to see that σ corresponds to that acting trivially on $\wedge^2 Z$ and by multiplication by -1 on $Sym^2 Z$. $\square$

Now Theorem 2.1.1 follows from the next proposition.

Proposition 2.1.6. *Let* Z *be a complex* $2n$*-dimensional space. The space* $Sym^n(\wedge^2 Z)$ *contains a unique* $GL(Z)$*-invariant subspace of codimension* 1.

Proof. First let us show the existence. Note that there exists a canonical map $Sym^n(\wedge^2 Z) \to \wedge^{2n} Z$ given by $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \wedge \cdots \wedge x_n$. Since $\wedge^{2n} Z$ is one dimensional the existence follows.

The representation of $GL(Z)$ in $Sym^n(\wedge^2 Z)$ is algebraic and hence completely reducible. Moreover the center of $GL(Z)$ acts by a multiplication by a character. Hence all one-dimensional components in $Sym^n(\wedge^2 Z)$ must be isomorphic. Thus the uniqueness follows from the following result due to Howe [24] (which also will be used later). $\square$

Proposition 2.1.7 (*Howe [24, p. 563, Proposition 2]*). *Let Z be an even-dimensional complex vector space. Let $k \geqslant 0$ be an integer. The natural representation of $GL(Z)$ in $Sym^k(\wedge^2 Z)$ is multiplicity free.*

Thus Theorem 2.1.1 is proved. $\square$

Now we will introduce a quaternionic analogue of the space of complex vectors of type (k,k). Let W be a right $\mathbb{H}$-module, $\dim_{\mathbb{H}} W = n$. The space $Sym^\bullet(W \otimes_{\mathbb{H}} \bar{W})^\sigma$ is a commutative associative algebra (it is full symmetric algebra of a real vector space). For $0 \leqslant k \leqslant n$ one has a map

$$Sym^k(W \otimes_{\mathbb{H}} \bar{W})^\sigma \times Sym^{n-k}(W \otimes_{\mathbb{H}} \bar{W})^\sigma \to M(W)$$

which is a composition of the product with the Moore map M. Let us denote by $L^k(W)$ the left kernel of this map, namely

$$L^k(W) = \{x \in Sym^k(W \otimes_{\mathbb{H}} \bar{W})^\sigma | \, M(x \cdot y) = 0 \forall y \in Sym^{n-k}(W \otimes_{\mathbb{H}} \bar{W})^\sigma\}.$$

Let us define $R^k(W) := Sym^k(W \otimes_{\mathbb{H}} \bar{W})^\sigma / L^k(W)$. Then we obviously have

Claim 2.1.8. *The pairing $R^k(W) \times R^{n-k}(W) \to M(W)$ is perfect. In particular one has an isomorphism $R^k(W) \simeq R^{n-k}(W)^* \otimes M(W)$ commuting with the action of $Aut_{\mathbb{H}} W$.*

Remind that the group $Aut_{\mathbb{H}}(W)$ of invertible automorphisms of W is isomorphic to $GL_n(\mathbb{H})$, and its complexification is isomorphic to $GL_{2n}(\mathbb{C})$.

Theorem 2.1.9 (*Irreducibility property*). *For $0 \leqslant k \leqslant n$ the space $R^k(W) \otimes_{\mathbb{R}} \mathbb{C}$ is an irreducible $Aut_{\mathbb{H}} W$-module. As $GL_{2n}(\mathbb{C})$-module it has highest weight $(\underbrace{1,\ldots,1}_{2k\ times}, \underbrace{0,\ldots,0}_{2(n-k)\ times})$.*

Proof. By Proposition 2.1.7 the $GL_{2n}(\mathbb{C})$-module $R^k(W) \otimes_{\mathbb{R}} \mathbb{C}$ is multiplicity free. Let $(\lambda_1 \geqslant \cdots \geqslant \lambda_{2n})$ be the highest weight of an irreducible component H_1 of this representation (thus $\lambda_i \in \mathbb{Z}$). Since it can be realized as a subrepresentation in the tensor power $Z^{\otimes 2n}$ (where $Z = Hom_{\mathbb{C}_{\mathbb{H}}}(T, {}^{\mathbf{C}}W)$) it corresponds to a Young diagram, and hence $\lambda_i \geqslant 0$ for all i. By Claim 2.1.8 there exists an irreducible submodule H_2 of $R^{n-k}(W)$ such that $H_1 \simeq H_2^* \otimes M(W)$. Let $(\mu_1 \geqslant \cdots \geqslant \mu_{2n})$ be the highest weight of H_2. Similarly $\mu_i \geqslant 0$. Then the highest weight of $H_2^* \otimes M(W)$ is equal to $(1-\mu_{2n}, \dots, 1-\mu_1)$. Thus we get

$$\lambda_i = 1 - \mu_{2n-i+1},\ i = 1, \dots, 2n.$$

Hence $\lambda_i \leqslant 1$ for all i. Thus the highest weight of H_1 is equal to $(\underbrace{1, \dots, 1}_{l \text{ times}}, \underbrace{0, \dots, 0}_{2n-l \text{ times}})$. The center of $GL_{2n}(\mathbb{C})$ consists of $\{\lambda \cdot Id \,|\, \lambda \in \mathbb{C}^*\}$, and it acts on $R^k(W) \otimes_{\mathbb{R}} \mathbb{C}$ by multiplication by λ^{2k}. Hence $l = 2k$. This proves Theorem 2.1.9. $\square$

Theorem 2.1.10. *The correspondence $W \mapsto R^k(W)$ is a functor from the category of finite-dimensional right $\mathbb{H}$-modules to the category of finite dimensional real vector spaces. Namely a morphism $W_1 \to W_2$ induces a morphism $R^k(W_1) \to R^k(W_2)$ in a way compatible with compositions of morphisms.*

Proof. Let $f : W_1 \to W_2$ be a morphism of right $\mathbb{H}$-modules. Then f induces a morphism of $\mathbb{R}$-vector spaces $f : Sym^k(W_1 \otimes \bar{W}_1)^\sigma \to Sym^k(W_2 \otimes \bar{W}_2)^\sigma$. We have to show that $f(L^k(W_1)) \subset L^k(W_2)$. Note that $Sym^k(W \otimes \bar{W})^\sigma \otimes_{\mathbb{R}} \mathbb{C}$ can be realized (functorially) as a submodule in $Z^{\otimes 2k}$ with $Z = Hom_{\mathbb{C}\mathbb{H}}(T, {}^{\mathbf{C}}W)$. Then $L^k(W)$ can be realized as a kernel of corresponding Young symmetrizer which does not depend on W. Hence $f(L^k(W_1)) \subset L^k(W_2)$. $\square$

Set $R(W) := \oplus_{k=0}^n R^k(W)$ where $n = \dim_{\mathbb{H}} W$.

Proposition 2.1.11. *$R(W)$ is a commutative associative graded algebra with Poincaré duality. Any morphism $f : W_1 \to W_2$ induces a homomorphism of algebras $f : R(W_1) \to R(W_2)$.*

Remark. Recall that the Poincaré duality means that $\dim_{\mathbb{R}} R^n(W) = 1$ and the pairing induced by the multiplication $R^k(W) \otimes R^{n-k}(W) \to R^n(W)$ is perfect.

Proof of Proposition 2.1.11. The only property one has to check is

$$L^k(W) \cdot Sym^l(W \otimes_{\mathbb{H}} \bar{W})^\sigma \subset L^{k+l}(W).$$

Let us check it. Take any $g \in Sym^{n-k-l}(W \otimes_{\mathbb{H}} \bar{W})^{\sigma}$. One has $M(L^k(W) \cdot Sym^l(W \otimes_{\mathbb{H}} \bar{W})^{\sigma} \cdot g) \subset M(L^k(W) \cdot Sym^{n-k}(W \otimes_{\mathbb{H}} \bar{W})^{\sigma}) = 0$. $\square$

Now we can define a quaternionic analogue of the space of translation invariant forms of type (k,k) on a complex space. Let V be a finite-dimensional right $\mathbb{H}$-module.

Definition 2.1.12. Define

$$\Omega^{k,k}(V) := R^k(\bar{V}^*),$$

$$\Omega^{\bullet}(V) := \oplus_{k=0}^{n} \Omega^{k,k}(V).$$

Theorem 2.1.13. *The correspondence $V \mapsto \Omega^{\bullet}(V)$ is a contravariant functor from the category of finite-dimensional right $\mathbb{H}$-modules to the category of finite-dimensional commutative associative graded algebras. For a fixed V the graded algebra $\Omega^{\bullet}(V)$ satisfies the Poincaré duality. For any k, $\Omega^{k,k}(V) \otimes_{\mathbb{R}} \mathbb{C}$ is an irreducible $(Aut_{\mathbb{H}} V \otimes_{\mathbb{R}} \mathbb{C})$-module.*

Proof. The proof immediately follows from Theorems 2.1.10, 2.1.9, and Claim 2.1.8. $\square$

Let us denote by $\mathcal{H}(V)$ the space of hyperhermitian forms on V.

Proposition 2.1.14. *There exists canonical isomorphism $\mathcal{H}(V) = \Omega^{1,1}(V)$.*

Proof. Let $B : V \times V \to \mathbb{H}$ be a hyperhermitian form. It defines a map $B_1 : V \to Hom_{\mathbb{H}}(V, \mathbb{H}) = V^*$. Let us identify V with $\bar{V}$ as $\mathbb{R}$-vector spaces. Thus $B_1 : \bar{V} \to V^*$. Since B is semi-linear with respect to the first argument, $B_1 : \bar{V} \to V^*$ is a morphism of left $\mathbb{H}$-modules. Thus $B_1 \in End_{\mathbb{H}}(\bar{V}, V^*) = (\bar{V})^* \otimes_{\mathbb{H}} V^* = \bar{V}^* \otimes_{\mathbb{H}} V^*$. The fact that B satisfies $B(x,y) = \overline{B(y,x)}$ means that $B_1 \in (\bar{V}^* \otimes_{\mathbb{H}} V^*)^{\sigma} = \Omega^{1,1}(V)$. Thus the above construction defines a canonical map $\mathcal{H}(V) \to \Omega^{1,1}(V)$. This is an isomorphism. $\square$

Proposition 2.1.15. (1) *Let V be a right $\mathbb{H}$-module, $n = \dim_{\mathbb{H}} V$. There exists a natural isomorphism*

$$(\Omega^{n,n}(V))^{\otimes 2} = \wedge_{\mathbb{R}}^{4n}(V^*).$$

(2) *Let $0 \to W \to V \to U \to 0$ be a short exact sequence of finite-dimensional right $\mathbb{H}$-modules of quaternionic dimensions k, n, l, respectively. Then there exists canonical isomorphism*

$$\Omega^{n,n}(V) = \Omega^{k,k}(W) \otimes_{\mathbb{R}} \Omega^{l,l}(U).$$

Proof. (1) Let us fix an $\mathbb{H}$-basis $e_1, \dots, e_n$ of V. Let $Id \in \mathcal{H}_n \simeq \mathcal{H}(V)$ be the identity matrix. Then $(Id)^n \in \Omega^{n,n}(V)$ spans $\Omega^{n,n}(V)$ over $\mathbb{R}$. Let us define isomorphism $\Omega^{n,n}(V) \tilde{\to} \wedge^{4n}_{\mathbb{R}} V^*$ such that $(Id)^n$ is mapped to $\wedge_{p=1}^n \left(e_p^* \wedge (e_p \cdot I)^* \wedge (e_p \cdot J)^* \wedge (e_p \cdot K)^*\right)$ where $*$ denotes taking the bi-dual basis. It is easy to see that this isomorphism is independent of a choice of a basis $e_1, \dots, e_n$ of V.

(2) Let us define a map $\Omega^{k,k}(W) \otimes_{\mathbb{R}} \Omega^{l,l}(U) \to \Omega^{n,n}(V)$ by $x \otimes y \mapsto x \cdot y$. It is easy to see that this map is well defined and is an isomorphism. □

Remark 2.1.16. Let $\dim_{\mathbb{H}} V = n$. The real line $\Omega^{n,n}(V)$ is canonically oriented. The orientation is defined as follows. Let us fix an arbitrary $\mathbb{H}$-basis $e_1, \dots, e_n$ of V. Then $\Omega^{1,1}(V)$ is identified with $\mathcal{H}(V) \simeq \mathcal{H}_n$ by Proposition 2.1.14. Let $Id \in \mathcal{H}_n$ be the identity matrix. Then $(Id)^n$ spans $\Omega^{n,n}(V)$. Let us choose the orientation of $\Omega^{n,n}(V)$ so that $(Id)^n$ is a positive element. It is easy to check that this orientation is independent of a choice of a basis of V.

2.2. *Positive forms*

In this subsection we will define convex cones of weakly and strongly positive forms in $\Omega^{k,k}(V)$. The exposition is analogous to the complex case as in Harvey [22] (see also Lelong [29]).

Let V be finite-dimensional right $\mathbb{H}$-module, $\dim_{\mathbb{H}} V = n$. First recall that by Remark 2.1.16 the space $\Omega^{n,n}(V)$ is a one-dimensional real vector space with canonical orientation. Thus we can define non-negative elements in $\Omega^{n,n}(V)$. This (closed) half line will be denoted by $\Omega^{n,n}(V)_{\geqslant 0}$.

Definition 2.2.1. (1) An element $\eta \in \Omega^{k,k}(V)$ is called *strongly positive* if it can be presented as a finite sum of elements of the form $f^*\xi$ where $f : V \to U$ is a morphism of right $\mathbb{H}$-modules, $\dim_{\mathbb{H}} U = k$, $\xi \in \Omega^{k,k}(U)_{\geqslant 0}$.

(2) An element $\eta \in \Omega^{k,k}(V)$ is called *weakly positive* (or just positive) if for any strongly positive element $\zeta \in \Omega^{n-k,n-k}(V)$ the product $\eta \cdot \zeta \in \Omega^{n,n}(V)_{\geqslant 0}$.

Clearly positive and weakly positive elements form convex cones. Let us denote by $C^k(V)$ (resp. $K^k(V)$) the cone of strongly (resp. weakly) positive elements in $\Omega^{k,k}(V)$.

Proposition 2.2.2. *Any strongly positive element of $\Omega^{k,k}(V)$ is weakly positive, namely $C^k(V) \subset K^k(V)$.*

Proof. It is sufficient to prove the following statement. Let $V = U \oplus W$, $\dim_{\mathbb{H}} U = k$, $\dim_{\mathbb{H}} W = l$. Let $p_U : V \to U$, $p_W : V \to W$ be the corresponding projections. Let $\eta \in \Omega^{k,k}(U)_{\geqslant 0}$, $\zeta \in \Omega^{l,l}(V)_{\geqslant 0}$. Then one should check that $p_U^*\eta \cdot p_W^*\zeta \in \Omega^{n,n}(V)_{\geqslant 0}$. This is clear. □

The proof of the next proposition is obvious.

Proposition 2.2.3.

$$C^k(V)\cdot C^l(V)\subset C^{k+j}(V).$$

The complex version of the next result is due to Lelong [29].

Proposition 2.2.4. *For $k=0,1,n-1,n$ the cones of weakly and strongly positive elements coincide.*

Proof. For $k=0,n$ the statement is obvious. Let us prove it for $k=1$. We may assume that $V=\mathbb{H}^n$. Let us fix on $\mathbb{H}^n$ the standard hyperhermitian form $\sum_{i=1}^n |q_i|^2$. Then as we have noticed in Section 2.1 the space $W\otimes_{\mathbb{H}}\bar W$ $(W=\bar V^*)$ can be identified with the space of hyperhermitian matrices $\mathcal{H}_n$. Let us check that $L^1(W)=0$. It is enough to show that for any $A\in\mathcal{H}_n$, $A\neq 0$ there exists $B\in\mathcal{H}_n$ such that the mixed Moore determinant $\det(A,B,\dots,B)\neq 0$. By Claim 1.2.7 we may assume that A is diagonal. Say $A=diag(t_1,\dots,t_n)$, $t_1\neq 0$. Take $B=diag(0,1,\dots,1)$. Then $\det(A,B,\dots,B)=t_1\neq 0$. Thus $\Omega^{1,1}(V)$ is identified with $\mathcal{H}_n$. Let $A\in K^1(V)$. We may assume that A is diagonal, $A=diag(t_1,\dots,t_n)$. It is easy to see that $t_i\geqslant 0$ for any i. This implies that $A\in C^1(V)$. Hence $C^1(V)=K^1(V)$.

Let us assume that $k=n-1$. We may and will assume that $V=\mathbb{H}^n$. Let us fix on $\mathbb{H}^n$ the standard hyperhermitian form $\sum_{i=1}^n |q_i|^2$. This form gives an identification $\Omega^{n,n}(\mathbb{H}^n)\simeq\mathbb{R}$. By Theorem 2.1.13 we have $\Omega^{n-1,n-1}(\mathbb{H}^n)=(\Omega^{1,1}(\mathbb{H}^n))^*\otimes\Omega^{n,n}(\mathbb{H}^n)=(\Omega^{1,1}(\mathbb{H}^n))^*=(\mathcal{H}_n)^*$. However on $\mathcal{H}_n$ we have a non-degenerate bilinear form

$$(X,Y):=Tr(X\cdot Y),$$

where for a quaternionic matrix $Z=(z_{ij})$ by definition $Tr(Z):=Re\sum_i z_{ii}$. Thus using this form one can identify $\mathcal{H}_n^*=\mathcal{H}_n$. One can easily see that with this identification the cone $\mathcal{C}\subset\mathcal{H}_n$ of non-negative definite hyperhermitian matrices is self-dual, namely $\mathcal{C}=\{X\in\mathcal{C}\mid Tr(X\cdot Y)\geqslant 0\,\forall Y\in\mathcal{C}\}$.

Thus we have an identification

$$\Omega^{n-1,n-1}(\mathbb{H}^n)=\mathcal{H}_n. \tag{3}$$

It is easy to see that an element $\eta\in\Omega^{n-1,n-1}(\mathbb{H}^n)$ is strongly positive if and only if it can be represented as a sum with non-negative coefficients of elements of the form $p^*(vol^{1/2})$ where $p\colon\mathbb{H}^n\to L$ is an *orthogonal* projection onto a quaternionic subspace L of quaternionic dimension $n-1$, and $vol^{1/2}\in\Omega^{n-1,n-1}(L)$ is the positive square root of the Lebesgue measure on L induced by the metric. Let us fix such L. Making an orthogonal transformation we may assume that L is spanned by the last $n-1$

coordinates. We claim that under the above identification (3) the element η corresponds to the matrix $\frac{1}{n}S$ where $S = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$. Let us check it. We have to show that for any $X = (x_{ij}) \in \mathcal{H}_n$ one has

$$\det(X, p^*(I_{n-1}), \dots, p^*(I_{n-1})) = Tr(X \cdot \frac{1}{n} S).$$

Note first that $vol^{1/2} = (I_{n-1})^{n-1} \in \Omega^{n-1,n-1}(\mathbb{H}^n)$ where I_{n-1} is the identity matrix of size $n-1$. Obviously

$$p^*(I_{n-1}) = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{n-1} \end{array} \right]. \tag{4}$$

We have

$$\begin{aligned} \det(X, p^*(I_{n-1}), \dots, p^*(I_{n-1})) &= \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1 \dots \partial \lambda_n} \det(\lambda_1 X + \sum_{i=2}^n \lambda_i p^* I_{n-1}) \\ &= \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1 \dots \partial \lambda_n} \det(\lambda_1 X + (\sum_{i=2}^n \lambda_i) p^* I_{n-1}). \end{aligned}$$

Using identities (2) and (4) we easily get

$$\begin{aligned} &\det(X, p^*(I_{n-1}), \dots, p^*(I_{n-1})) \\ &\quad = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1 \dots \partial \lambda_n} \sum_{I \subset \{2,\dots,n\}} (\sum_{i=2}^n \lambda_i)^{|I|} \det M_I(\lambda_1 X) \\ &\quad = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1 \dots \partial \lambda_n} (\sum_{i=2}^n \lambda_i)^{n-1} \lambda_1 x_{11} = \frac{1}{n} x_{11} = Tr(X \cdot \frac{1}{n} S). \end{aligned}$$

Hence matrices of the form $p^*(vol^{1/2})$, where p is an orthogonal projection onto a quaternionic subspace of quaternionic dimension $n-1$, are precisely (hyperhermitian) matrices which can be written at some orthonormal basis is the form $\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$. The set of linear combinations of such matrices with non-negative coefficients coincides with the cone $\mathcal{C}$ on non-negative definite matrices. Hence under identification (3) the cone $C^{n-1}(\mathbb{H}^n) \subset \Omega^{n-1,n-1}(\mathbb{H}^n)$ corresponds to the cone $\mathcal{C} \subset \mathcal{H}_n$. As we have mentioned the cone $\mathcal{C}$ is self-dual. Hence $K^{n-1}(\mathbb{H}^n) = C^{n-1}(\mathbb{H}^n)$. This proves Proposition 2.2.4. $\square$

Proposition 2.2.5. *The cones of strongly and weakly positive elements in $\Omega^{k,k}(V)$ have non-empty interiors.*

Proof. Since $C^k(V) \subset K^k(V)$ by Proposition 2.2.2, it is enough to check that the cone $C^k(V)$ of strongly positive elements in $\Omega^{k,k}(V)$ has non-empty interior. We will use the following well known and simple fact.

Lemma 2.2.6. *Let X be a finite-dimensional real vector space. Let C be a convex cone in X. Then C has a non-empty interior if and only if $C - C = X$ where $C - C := \{c_1 - c_2 \,|\, c_1, c_2 \in C\}$.*

Observe first that the cone $C^k(V)$ is $Aut_{\mathbb{H}}(V)$-invariant (as well as the cone $K^k(V)$). Note that $C^k(V) - C^k(V)$ is a non-zero $Aut_{\mathbb{H}}\, V$-invariant subspace of $\Omega^{k,k}(V)$. By the irreducibility property (Theorem 2.1.9) this subspace must be equal to $\Omega^{k,k}(V)$. Hence by Lemma 2.2.6 $C^k(V)$ has non-empty interior. $\square$

3. More quaternionic pluripotential theory

3.1. Positive currents

In this subsection we construct a quaternionic analogue of form-valued measures (currents) $dd^c f_1 \wedge \cdots \wedge dd^c f_k$ where f_i are continuous psh functions.

Let us start with some well known general remarks (see e.g. [20]). Let M be a smooth manifold. Let E be a finite-dimensional vector bundle over M. Let $C_c^\infty(M, E)$ denotes the Schwartz space of infinitely smooth compactly supported sections of E equipped with the standard topology. Let $|\omega_M|$ denote the line bundle of densities on M.

Definition 3.1.1. A generalized section of E is a continuous linear functional on $C_c^\infty(M, E^* \otimes |\omega_M|)$.

The space of generalized sections is denoted by $C^{-\infty}(M, E)$. Recall also that $C^\infty(M, E) \subset C^{-\infty}(M, E)$. Indeed any $f \in C^\infty(M, E)$ defines a continuous linear functional on $C_c^\infty(M, E^* \otimes |\omega_M|)$ by $\langle f, \phi \rangle = \int_M \phi(f)$.

Let now X be a finite-dimensional real vector space. The space of smooth (resp. generalized) functions on M with values in X is denoted by $C^\infty(M, X)$ (resp. $C^{-\infty}(M, X)$).

Let $K \subset X$ be a closed convex cone in X. Recall that the polar of K is denoted by

$$K^\circ := \{y \in X^* |\, \langle y, x \rangle \geqslant 0 \ \forall x \in K\}.$$

Then K° is a closed convex cone in X^*.

Definition 3.1.2. A *K°-valued smooth density* on M is a section $\mu \in C^{\in} fty(M, X^* \otimes |\omega_M|)$ such that its value at any point $p \in M$ has the form

$$\mu(p) = y \otimes l,$$

where $y \in K^\circ$, $l \in |\omega_M|_p$, $l \geqslant 0$.

It is easy to see that K°-valued smooth densities form a convex cone in $C^\infty(M, X^* \otimes |\omega_M|)$.

Definition 3.1.3. A generalized K-valued function on M is an $\mathbb{R}$-linear continuous functional

$$f : C_c^\infty(M, X^* \otimes |\omega_M|) \to \mathbb{R}$$

such that $f(\phi) \geqslant 0$ for any smooth compactly supported K°-valued density ϕ on M.

Proposition 3.1.4. *Assume that $K^\circ \subset X^*$ has non-empty interior. Let f be a generalized K-valued function on M. Then f has order* 0, *i.e. f is a continuous $\mathbb{R}$-linear functional on $C_c(M, X^* \otimes |\omega_M|)$ which takes non-negative values on K°-valued smooth densities.*

Proof. Let us fix a smooth density $m \in C^\infty(M, |\omega_M|)$ which is strictly positive at each point. Let f be a generalized K-valued function. Fix a compact subset A of M. We have to show that there exists a constant C_A such that for any $\phi \in C_c^\infty(M, X^* \otimes |\omega_M|)$ with $supp(\phi) \subset A$ one has

$$|\langle f, \phi \rangle| \leqslant C_A ||\phi||_0.$$

Let us fix a function $\gamma \in C_c^\infty(M, \mathbb{R})$ which is equal to 1 on A and $\gamma \geqslant 0$ on M. Let us fix a vector ξ from the interior on K°. There exists a constant C'_A such that for any $\phi \in C_c^\infty(X, X^* \otimes |\omega_M|)$ with $supp\, \phi \subset A$ the function $\psi := \phi + C'_A ||\phi||_0 \xi \cdot m \cdot \gamma$ is a K°-valued density. Thus

$$f(\phi) = f(\psi - C'_A ||\phi||_0 \xi \cdot m \cdot \gamma) \geqslant -\, ||\phi||_0 f(C'_A \xi \cdot m \cdot \gamma) = -C_A ||\phi||_0.$$

Replacing ϕ by $-\phi$ we obtain the inverse inequality, and hence

$$|f(\phi)| \leqslant C_A ||\phi||_0. \qquad \square$$

We will apply the above constructions in the following situation. Let V be a right $\mathbb{H}$-module, $\dim_{\mathbb{H}} V = n$. Fix an integer $0 \leqslant k \leqslant n$. Take $X = \Omega^{k,k}(V)$. Then by Claim 2.1.8 one can canonically identify X^* with $\Omega^{n-k,n-k}(V) \otimes (\Omega^{n,n}(V))^*$. Let us take the convex cone $K \subset X = \Omega^{k,k}(V)$ to be the cone of weakly positive elements (in sense of Definition 2.2.1). Recall that K and K° have non-empty interiors by Proposition 2.2.5.

For an open subset $\mathcal{O} \subset V$, let us call by *currents* the generalized functions on $\mathcal{O}$ with values in $\Omega^\bullet(V)$.

Definition 3.1.5. Let $\mathcal{O} \subset V$ be an open subset. A current $f \in C^{-\infty}(\mathcal{O}, \Omega^{k,k}(V))$ is called (*weakly*) *positive* if f is a generalized K-valued function on $\mathcal{O}$ in sense of Definition 3.1.1.

It follows from Proposition 3.1.4 that any positive current has order zero. Now we are going to construct a quaternionic analogue of the expression $dd^c f$. We construct a linear differential operator

$$D_2 : C^2(\mathcal{O}, \mathbb{R}) \to C(\mathcal{O}, \Omega^{1,1}) = C(\mathcal{O}, \mathcal{H}(V)).$$

Fix any point $v \in V$. The Hessian $B := Hess_v F$ of f at v defines a quadratic form on ${}^{\mathbf{R}}V$. Let us extend it to $V \otimes_{\mathbb{R}} \mathbb{H}$ as a hyperhermitian form B' by the rule $B'(x \otimes q_1, y \otimes q_2) = \bar{q}_1 B(x, y) q_2$. Define the subspace

$$V' := \{X - X \cdot I \otimes i - X \cdot J \otimes j - X \cdot K \otimes k \mid X \in V\} \subset V \otimes_{\mathbb{R}} \mathbb{H}.$$

Then V and V' are naturally isomorphic as right $\mathbb{H}$-modules.

Consider the restriction of B' to $V' \subset V \otimes_{\mathbb{R}} \mathbb{H}$. Then $B'|_{V'}$ is a hyperhermitian form on V', and in coordinates it is given by the matrix $\left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}\right)$. This defines the map we need

$$D_2 : C^2(\mathcal{O}, \mathbb{R}) \to C(V, \Omega^{1,1}(V)).$$

Note that if $f \in C^2(\mathcal{O}) \cap P(V)$ then $D_2 f \in \Omega^{1,1}(V)$ is non-negative pointwise. Hence for such f, $D_2 f$ is a non-negative current.

Now if $k \leqslant n$ and $f_1, \dots, f_k \in C^2(V, \mathbb{R})$ then we can consider $D_2 f_1 \cdot \dots \cdot D_2 f_k \in C(\mathcal{O}, \Omega^{k,k}(V))$. Note also that by Proposition 2.2.3 this is a non-negative current provided all f_i are psh. We want to define this expression as a current for all $f_i \in C(\mathcal{O}) \cap P(\mathcal{O})$.

Theorem 3.1.6. *Let $\mathcal{O} \subset V$ be an open subset. For any k-tuple of functions $f_1, \dots, f_k \in C(\mathcal{O}) \cap P(\mathcal{O})$ one can define a non-negative $\Omega^{k,k}(V)$-valued current denoted by $D_2 f_1 \cdot \dots \cdot D_2 f_k$ which is uniquely characterized by the following properties*:

1. *if $f_1, \dots, f_k \in C^2(\mathcal{O})$ then this current coincides with the defined above*;
2. *if sequences $\{f_i^{(N)}\} \subset C(\mathcal{O}) \cap P(\mathcal{O})$, $i = 1, \dots, k$, converge to $\{f_i\}$ in C^0-topology, i.e. uniformly on compact subsets of $\mathcal{O}$, then*

$$D_2 f_1^{(N)} \cdot \dots \cdot D_2 f_k^{(N)} \to D_2 f_1 \cdot \dots \cdot D_2 f_k \text{ as } N \to \infty$$

weakly, i.e. in $(C^0)^*$-topology.

Proof. In [5] we have defined a non-negative measure $(D_2 f)^n$ for any function $f \in C(\mathcal{O}) \cap P(\mathcal{O})$ (in the notation of [5] $(D_2 f)^n$ is identified with $\det\left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}\right)$) such that if a sequence $\{f^{(N)}\} \subset C(\mathcal{O}) \cap P(\mathcal{O})$ converges to f in $C^0(\mathcal{O})$-topology then $(D_2 f^{(N)})^n \to (D_2 f)^n$ weakly. Linearizing the expression $(D_2 f)^n$ we define non-negative measures $D_2 f_1 \cdot \dots \cdot D_2 f_n$ for any n-tuple $f_1, \dots, f_n \in C(\mathcal{O}) \cap P(\mathcal{O})$, and it satisfies the same continuity property.

Let $1 \leqslant k < n$. We have to define non-negative currents $D_2 f_1 \cdot \dots \cdot D_2 f_k$ for $f_1, \dots, f_k \in C(\mathcal{O}) \cap P(\mathcal{O})$. Let us fix a basis in V. Let us fix $\mathcal{H}(V)$-valued functions $A_1, \dots, A_{n-k} \in C(\mathcal{O}, \mathcal{H}(V))$. Note also that we can and will identify $\mathcal{H}(V)$ with the space of hyperhermitian matrices $\mathcal{H}_n$. Let us denote for brevity by $\partial^2 u$ the matrix $\left(\frac{\partial^2 u}{\partial q_i \partial \bar{q}_j}\right)$. We have to define measures $\det(\partial^2 f_1, \dots, \partial^2 f_k, A_1, \dots, A_{n-k})$ on V which have the obvious meaning for $f_i \in C^2(V)$. We will do it by induction in n. For $n = 1$ the measure coincides with the usual Laplacian Δf_1. Assume that $n > 1$. For $f_i \in C^2(\mathcal{O})$ the expression $\det(\partial^2 f_1, \dots, \partial^2 f_k, A_1, \dots, A_{n-k})$ is linear with respect to each A_j. But every $\mathcal{H}_n$-valued function can be presented as a finite linear combination of matrices such that in appropriate coordinate system each of the summand has the form $\phi \cdot \begin{bmatrix} 0 & \dots & 0 & 0 \\ & \ddots & & \\ 0 & \dots & 0 & 1 \end{bmatrix}$ where $\phi \in C(\mathcal{O}, \mathbb{R})$. Thus we may assume that A_1 has such a form. Then for any $f_1, \dots, f_k \in C^2(\mathcal{O})$ one has by Lemma 1.2.16

$$\begin{aligned} &\det(\partial^2 f_1, \dots, \partial^2 f_k, A_1, \dots, A_k) \\ &\quad = \frac{1}{n} \phi \cdot \det{}_{n-1}(M_{\{n\}}(\partial^2 f_1), \dots, M_{\{n\}}(\partial^2 f_k), M_{\{n\}}(A_2), \dots, M_{\{n\}}(A_{n-k})), \end{aligned}$$

where as previously $M_{\{n\}}(X)$ denotes the minor of the matrix X obtained by deleting the last row and the last column. Inductively we are reduced to the following situation. Fix a decomposition $\mathbb{H}^n = \mathbb{H}^k \oplus \mathbb{H}^{n-k}$. For a k-tuple of functions $f_1, \dots, f_k \in C(\mathcal{O}) \cap P(\mathcal{O})$ we want to define measures on $\mathcal{O}$ denoted by $\det_k([\partial^2 f_1]_k, \dots, [\partial^2 f_k]_k)$ which depend continuously (in the sense of the weak convergence of measures) in $f_i \in C(\mathcal{O}) \cap P(\mathcal{O})$. Here $[X]_k$ denotes the $(k \times k)$-matrix obtained from X by deleting the last $(n-k)$ rows and columns. For $\psi \in C_c(\mathcal{O})$ we will define the integral

$$\int_{\mathbb{H}^n} \psi \det{}_k([\partial^2 f_1]_k, \dots, [\partial^2 f_k]_k) =: (*)$$

and show that it is continuous with respect to $f_1, \dots, f_k \in C(\mathcal{O}) \cap P(\mathcal{O})$. Note that if $f_i \in C^2(\mathcal{O})$ for all i then

$$(*) = \int_{\mathbb{H}^{n-k}} d\, vol [\int_{\mathbb{H}^k} \psi \det{}_k([\partial^2 f_1]_k, \dots, [\partial^2 f_k]_k) d\, vol].$$

By Theorem 1.3.3 the inner integral is defined for all $f_1, \dots, f_k \in C(\mathcal{O}) \cap P(\mathcal{O})$ and depends continuously on f_i's. Moreover Lemma 1.2.17 implies the following estimate:

$$|\int_{\mathbb{H}^k} \psi \det{}_k([\partial^2 f_1]_k, \dots, [\partial^2 f_k]_k) d\, vol| \leqslant \int_{\mathbb{H}^k} |\psi| \det{}_k([\sum_{i-1}^{k} \partial^2 f_i]_k) d\, vol.$$

By Lemma 2.1.9 in [5] for $f \in C(\mathcal{O}) \cap P(\mathcal{O})$ one has

$$\int_{\mathbb{H}^k} |\psi| \det(\partial^2 f) d\, vol \leqslant C_k ||\psi||_{L^\infty(\mathbb{H}^n)} ||f||^k_{L^\infty(U)},$$

where U is a compact neighborhood of $supp(\psi)$ and C_k is a constant depending on k only. Hence

$$|\int_{\mathbb{H}^k} \psi \det{}_k([\partial^2 f_1]_k, \dots, [\partial^2 f_k]_k) d\, vol| \leqslant C'_k ||\psi||_{L^\infty(\mathbb{H}^n)} (\max{}_i ||f_i||_{C(U)})^k.$$

This estimate and the continuity with respect to f_i's of the inner integral in $(*)$ imply that $(*)$ is defined for all $f_i \in C(\mathcal{O}) \cap P(\mathcal{O})$ (since any $f \in C(\mathcal{O}) \cap P(\mathcal{O})$ can be approximated in C^0-topology by functions from $C^2(\mathcal{O}) \cap P(\mathcal{O})$) and $(*)$ depends continuously on f_i's. $\square$

3.2. Quaternionic Blocki's formula

Let $\mathcal{O}$ be an open subset in a right $\mathbb{H}$-vector space V, $\dim_{\mathbb{H}} V = n$. The complex version of the next result was proved by Blocki in [12].

Theorem 3.2.1. *Let $u, v \in P(\mathcal{O}) \cap C(\mathcal{O})$. Let $2 \leqslant p \leqslant n$. Then*

$$(D_2 \max\{u, v\})^p = (D_2 \max\{u, v\}) \cdot \sum_{k=0}^{p-1} (D_2 u)^k (D_2 v)^{p-1-k} - \sum_{k=1}^{p-1} (D_2 u)^k (D_2 v)^{p-k}. \tag{5}$$

Proof. First consider the case $p = 2$. By continuity of both sides in (5) we may assume that u, v are smooth. Let $\chi : \mathbb{R} \to [0, \infty)$ be a smooth function such that $\chi(x) = 0$

if $x \leqslant -1$, $\chi(x) = x$ if $x \geqslant 1$, and $0 \leqslant \chi' \leqslant 1$, $\chi'' \geqslant 0$ everywhere. Define

$$\psi_j := v + \frac{1}{j}\,\chi(j(u-v)),$$

$$\alpha := u - v,$$

$$w := \max\{u, v\}.$$

It is easy to see that $\psi_j \downarrow w$ uniformly on compact subsets and monotonically as $j \to \infty$.

Lemma 3.2.2.

$$\left(\frac{\chi(j\alpha)}{j}\right)_{p\bar{q}} = \chi'(j\alpha) \cdot \alpha_{p\bar{q}} + j\chi''(j\alpha)\alpha_{\bar{q}}\alpha_p.$$

Proof. Let us denote

$$e_0 = 1, e_1 = i, e_2 = j, e_3 = k.$$

$$\begin{aligned}
\left(\frac{\chi(j\alpha)}{j}\right)_{p\bar{q}} &= \frac{1}{j}\sum_{l,m=0}^{3} e_l(\chi(j\alpha))_{x_q^l x_p^m}\bar{e}_m \\
&= \sum_{l,m=0}^{3} e_l\left(\chi'(j\alpha)\cdot\alpha_{x_q^l}\right)_{x_p^m}\bar{e}_m \\
&= \sum_{l,m=0}^{3} e_l\left(j\chi''(j\alpha)\cdot\alpha_{x_q^l}\alpha_{x_p^m} + \chi'(j\alpha)\alpha_{x_q^l x_p^m}\right)\bar{e}_m \\
&= \chi'(j\alpha)\alpha_{p\bar{q}} + j\chi''(j\alpha)\alpha_{\bar{q}}\alpha_p. \qquad \square
\end{aligned}$$

Thus from Lemma 3.2.2 we obtain

$$\begin{aligned}
(\psi_j)_{p\bar{q}} &= v_{p\bar{q}} + \chi'(j\alpha)(u-v)_{p\bar{q}} + j\chi''(j\alpha)\alpha_{\bar{q}}\alpha_p \\
&= \chi'(j\alpha)u_{p\bar{q}} + (1-\chi'(j\alpha))v_{p\bar{q}} + j\chi''(j\alpha)\alpha_{\bar{q}}\alpha_p.
\end{aligned}$$

Since $0 \leqslant \chi' \leqslant 1$ and $\chi'' \geqslant 0$ this implies that ψ_j is psh. From the definition of ψ_j we have

$$(D_2\psi_j)^2 = (D_2 v)^2 + 2(D_2 v)\left(D_2\left(\frac{\chi(j\alpha)}{j}\right)\right) + \left(D_2\left(\frac{\chi(j\alpha)}{j}\right)\right)^2. \tag{6}$$

We have weak convergence

$$(D_2\psi_j)^2 \to (D_2 w)^2, \tag{7}$$

$$D_2 v \cdot D_2\left(\frac{\chi(j\alpha)}{j}\right) \to D_2(w-u)\cdot D_2 v. \tag{8}$$

Let us study the last term in (6), namely $\left(D_2\left(\frac{\chi(j\alpha)}{j}\right)\right)^2$.

Lemma 3.2.3. *Let* $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{H}^n$. *Then in* $\Omega^{2,2}(\mathbb{H}^n)$ *one has* $(AA^*)^2 = 0$.

Proof. One has to check that for all $X_1, \dots, X_{n-2} \in \mathcal{H}_n$

$$\det(AA^*, AA^*, X_1, \dots, X_{n-2}) = 0.$$

Multiplying A by an invertible matrix one may assume that $A = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Then $AA^* = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$. The result now follows from Lemma 1.2.16. $\square$

From Lemmas 3.2.2 and 3.2.3 one gets in $\Omega^{2,2}(V)$:

$$\left(\left(\frac{\chi(j\alpha)}{j}\right)_{p\bar{q}}\right)^2 = \left(\chi'(j\alpha)^2(\alpha_{p\bar{q}}) + 2j\chi'(j\alpha)\chi''(j\alpha)(\alpha_{\bar{q}}\alpha_p)\right)\cdot(\alpha_{p\bar{q}}). \tag{9}$$

Let $\gamma\colon \mathbb{R} \to \mathbb{R}$ be such that $\gamma' = (\chi')^2$. Then we have

Lemma 3.2.4.

$$\left(D_2\left(\frac{\chi(j\alpha)}{j}\right)\right)^2 = D_2\left(\frac{\gamma(j\alpha)}{j}\right)\cdot D_2\alpha.$$

Let us postpone the proof of Lemma 3.2.4, and finish the proof of Theorem 3.2.1 for $p=2$. One can choose γ so that $\gamma(-1)=0$. Then $\frac{\gamma(jx)}{j}\downarrow \max\{0,x\}$ uniformly on compact subsets and monotonically as $j\to\infty$, and

$$\left(D_2\left(\frac{\chi(j\alpha)}{j}\right)\right)^2 \to D_2(w-v)\cdot D_2\alpha \text{ weakly.}$$

This and (6)–(8) imply

$$\begin{aligned}(D_2w)^2 &= (D_2v)^2+2D_2(w-v)\cdot D_2v+D_2(w-v)\cdot D_2(u-v)\\ &= D_2w\cdot(D_2u+D_2v)-D_2u\cdot D_2v.\end{aligned}$$

This implies Theorem 3.2.1 for $p=2$. It remains to prove Lemma 3.2.4.

Proof of Lemma 3.2.4. We have

$$\begin{aligned}\left(\frac{\gamma(j\alpha)}{j}\right)_{p\bar q} &= \gamma'(j\alpha)\alpha_{p\bar q}+\alpha_{\bar q}\cdot(\gamma'(j\alpha))_p\\ &= (\chi'(j\alpha))^2\alpha_{p\bar q}+2j\chi'(j\alpha)\chi''(j\alpha)\alpha_{\bar q}\cdot\alpha_p.\end{aligned}$$

This and (9) imply Lemma 3.2.4. □

It remains to prove Theorem 3.2.1 for $p>2$. Set

$$x := D_2u,\ y := D_2u,\ z := D_2\max\{u,v\}. \tag{10}$$

One has

$$z^2 = z(x+y)-xy. \tag{11}$$

One has to show that

$$z^p = z\sum_{k=0}^{p-1} x^k y^{p-1-k} - \sum_{k=1}^{p-1} x^k y^{p-k}.$$

Let us prove it by induction in p. For $p = 2$ it is already proved. Assume that the statement is proved for p. Let us prove it for $p + 1$. By the assumption of induction one has

$$\begin{aligned} z^{p+1} &= z(z \sum_{k=0}^{p-1} x^k y^{p-1-k} - \sum_{k=1}^{p-1} x^k y^{p-k}) \\ &= (z(x+y) - xy) \sum_{k=0}^{p-1} x^k y^{p-1-k} - z \sum_{k=1}^{p-1} x^k y^{p-k} \\ &= z \sum_{k=0}^{p} x^k y^{p-k} - \sum_{k=1}^{p} x^k y^{p+1-k}. \qquad \square \end{aligned}$$

Theorem 3.2.5. *Let* $u, v \in C(\Omega) \cap P(\Omega)$. *Assume that* $\min\{u, v\} \in P(\Omega)$. *Then for* $1 \leqslant p \leqslant n$ *one has*

$$(D_2 u)^p + (D_2 v)^p = (D_2 \min\{u, v\})^p + (D_2 \max\{u, v\})^p.$$

Proof. Note that $\min\{u, v\} = u + v - \max\{u, v\}$. Hence

$$D_2 \min\{u, v\} = x + y - z$$

in notation (10). Thus we have to show that

$$x^p + y^p = z^p + (x + y - z)^p.$$

Let us prove it be the induction in p. For $p = 1$ the statement is clear. Let us assume that the statement is true for $p - 1$. Let us prove it for p. We have

$$\begin{aligned} (x + y - z)^p &= (x + y - z)(x^{p-1} + y^{p-1} - z^{p-1}) \\ &= (x^p + y^p - z^p) + [2z^p + xy^{p-1} - xz^{p-1} + yx^{p-1} \\ &\quad - yz^{p-1} - zx^{p-1} - zy^{p-1}]. \end{aligned}$$

Let us denote the summand in square brackets by A. By Theorem 3.2.1 and the assumption of induction we have

$$A = 2[z \sum_{k=0}^{p-1} x^k y^{p-1-k} - \sum_{k=1}^{p-1} x^k y^{p-k}] + xy^{p-1}$$

$$
\begin{aligned}
& -(x+y)[z\sum_{k=0}^{p-2} x^k y^{p-2-k} - \sum_{k=1}^{p-2} x^k y^{p-1-k}] \\
& +yx^{p-1} - z(x^{p-1}+y^{p-1}) \\
= {} & z\{2\sum_{k=0}^{p-1} x^k y^{p-1-k} - (x+y)\sum_{k=0}^{p-2} x^k y^{p-2-k} - (x^{p-1}+y^{p-1})\} \\
& -2\sum_{k=1}^{p-1} x^k y^{p-k} + xy^{p-1} + (x+y)\sum_{k=1}^{p-2} x^k y^{p-1-k} + yx^{p-1} = 0.
\end{aligned}
$$

This implies Theorem 3.2.5. □

4. Valuations and complex and quaternionic analysis

In this section we discuss some applications of results from complex and quaternionic pluripotential theory discussed in Section 3 to construction of continuous valuations. In Section 4.1 we discuss complex case, namely Kazarnovskii's pseudovolume and its generalizations. In Section 4.2 we discuss two quaternionic versions of Kazarnovskii's pseudovolume and generalizations.

4.1. Kazarnovskii's pseudovolume

Let $V = \mathbb{C}^n$ be a hermitian space. Let us consider on the Grassmannian $Gr_n(V)$ of *real* n-dimensional subspaces the function f such that for any $L \in Gr_n(V)$ $f(L)$ is equal to the coefficient of the area distortion under the orthogonal projection from L to $iL^{\perp}$. Sometimes $f(L)$ is denoted as $|\cos(L, iL^{\perp})|$ (the absolute value of the cosine of the angle between L and $iL^{\perp}$). Note also that f is proportional to $L \mapsto vol_{2n}(Q_L + i \cdot Q_L)$ where Q_L denotes the unit cube in L as previously.

Let $\mathcal{P}(V)$ denote the class of convex compact polytopes in V. For a polytope $P \in \mathcal{P}(V)$ let $\mathcal{F}_k(P)$ denote the set of k-dimensional faces of P. For a face $F \in \mathcal{F}_k(P)$ let $\gamma(F)$ denote the measure of the exterior angle at F. Let $\bar{F}$ denote the only k-dimensional linear subspace parallel to F.

Consider the valuation ϕ_f on $\mathcal{P}(\mathbb{C}^n)$ defined by

$$\phi_f(P) = \sum_{F \in \mathcal{F}_k(P)} f(\bar{F}) vol_n(F)\gamma(F).$$

This valuation will be called Kazarnovskii's pseudovolume since it was introduced and studied by Kazarnovskii [26,27] in connection with counting of zeros of exponential sums. From the integral geometric point of view Kazarnovskii's pseudovolume was studied in [7]. The first part of following result is due to Kazarnovskii [26], the second part was proved in [7].

Theorem 4.1.1 (*Kazarnovskii [26]*). *The Kazarnovskii pseudovolume admits an extension by continuity to $\mathcal{K}(\mathbb{C}^n)$. This extension is a continuous translation invariant n-homogeneous valuation on $\mathcal{K}(\mathbb{C}^n)$ which is $U(n)$-invariant.*

We will discuss below the reason why it is true. Kazarnovskii has given several formulas for the pseudovolume. Let us start with the following one. For a convex set $\mathcal{K}(\mathbb{C}^n)$ let us denote by h_K its supporting functional. Recall its definition:

$$h_K(x) = \sup_{y \in K} (x, y).$$

Theorem 4.1.2 (*Kazarnovskii [26]*). *Kazarnovskii's pseudovolume $P(K)$ is equal to*

$$\frac{1}{\kappa_n} \int_B (dd^c h_K)^n = \frac{2^{2n} n!}{\kappa_n} \int_B \det\left(\frac{\partial^2 h_K}{\partial z_i \partial \bar{z}_j}\right) d\,vol$$

where B is the unit ball in $\mathbb{C}^n$ and κ_n is the volume of the unit n-dimensional ball.

In this theorem the expression under the integral is understood as a measure (i.e. in the generalized sense). Thus the continuity of Kazarnovskii's pseudovolume follows from Theorem 1.1.3. We have also the following result.

Theorem 4.1.3. *Let $0 \leqslant k \leqslant n$. Fix $\psi \in C_0(\mathbb{C}^n, \Omega^{n-k,n-k})$. Then $K \mapsto \int_{\mathbb{C}^n} (dd^c h_K)^k \wedge \psi$ defines a continuous translation invariant k-homogeneous valuation on $\mathcal{K}(\mathbb{C}^n)$.*

In this paper we will prove a quaternionic analogue of this result, Theorem 4.2.1. The proof of Theorem 4.1.3 can be obtained similarly using the corresponding results from complex analysis. We omit the details of the proof of this theorem, and prove instead the quaternionic version.

4.2. Valuations on quaternionic spaces

Let $V = \mathbb{H}^n$ be the right quaternionic hyperhermitian space. We also fix the standard hyperhermitian product $(x, y) = \sum_{i=1}^n \bar{x}_i y_i$. The next result is a quaternionic version of Theorem 4.1.3.

Theorem 4.2.1. *Let $0 \leqslant k \leqslant n$. Fix $\psi \in C_0(V, \Omega^{n-k,n-k}(V) \otimes \Omega^{n,n}(V))$. Then*

$$K \mapsto \int_V (D_2 h_K)^k \cdot \psi$$

is a translation invariant continuous k-homogeneous valuation on $\mathcal{K}(V)$.

Proof. Translation invariance is obvious. Continuity follows from Theorem 3.1.6. To prove the valuation property let us observe first that if $K = K_1 \cup K_2$ with $K_1, K_2, K \in \mathcal{K}(V)$ then

$$h_K = \max\{h_{K_1}, h_{K_2}\},\ h_{K_1 \cap K_2} = \min\{h_{K_1}, h_{K_2}\}.$$

Hence the result follows from Theorem 3.2.5. $\square$

From Theorem 4.2.1 we immediately get the following corollary which provides new examples of $Sp(n)Sp(1)$-invariant valuations on $\mathbb{H}^n$.

Corollary 4.2.2. *Let $0 \leqslant k \leqslant n$. Fix $\psi \in C_0(V, \Omega^{n-k,n-k}(V) \otimes \Omega^{n,n}(V))$ which is invariant under the group $Sp(n)Sp(1)$. Then*

$$K \mapsto \int_V (D_2 h_K)^k \cdot \psi$$

is a translation invariant $Sp(n)Sp(1)$-invariant continuous k-homogeneous valuation on $\mathcal{K}(V)$.

Let us define a *quaternionic pseudovolume Q* which will be a quaternionic version of Kazarnovskii's pseudovolume. Let us consider on the Grassmannian of *real n*-dimensional subspaces the following function:

$$f(L) = \sqrt{vol_{4n}(Q_L + Q_L \cdot i + Q_L \cdot j + Q_L \cdot k)},$$

where Q_L is the unit cube in L as previously. Consider the valuation Q on $\mathcal{P}(\mathbb{H}^n)$ defined by

$$Q(P) = \sum_{F \in \mathcal{F}_n(P)} f(\bar{F}) vol_n(F) \gamma(F).$$

Theorem 4.2.3. *The valuation Q extends by continuity to $\mathcal{K}(\mathbb{H}^n)$. This extension is a continuous translation invariant n-homogeneous $Sp(n)Sp(1)$-invariant valuation on $\mathcal{K}(\mathbb{H}^n)$. Moreover for any $P \in \mathcal{P}(\mathbb{H}^n)$*

$$Q(P) = \frac{1}{\kappa_{3n}} \int_D \det\left(\frac{\partial^2 h_P}{\partial q_i \partial \bar{q}_j}\right) d\, vol, \tag{12}$$

where D is the unit ball in $\mathbb{H}^n$ and κ_{3n} is the volume of the unit $3n$-dimensional ball.

Clearly it is enough to prove only equality (12). The rest follows from Theorem 4.2.1. In order to prove equality (12) we will prove a more precise statement. We will describe explicitly the measure $\det(\frac{\partial^2 h_P}{\partial q_i \partial \bar{q}_j}) d\, vol$. We need to introduce more notation.

Let us fix a polytope $P \in \mathcal{P}(V)$. For a face $F \subset P$ let us define

$$F^{\vee} := \{u \in V^* \mid u(v) = h_P(u)\, \forall v \in F\},$$

where h_P is the supporting function of P as previously. One can easily check the following properties:

(1) $F^{\vee}$ is a closed convex polyhedral cone;
(2) $\dim F + \dim F^{\vee} = \dim_{\mathbb{R}} V$;
(3) $F_1 \subset F_2$ if and only if $F_1^{\vee} \supset F_2^{\vee}$;
(4) $int\, F^{\vee} \cap int\, G^{\vee} = \emptyset$ for $F \neq G$ where *int* denotes the relative interior of a cone;
(5) if $int F^{\vee} \cap G^{\vee} \neq \emptyset$ then $F^{\vee} \subset G^{\vee}$;
(6) the union of $int F^{\vee}$ when F runs over all non empty faces of P is equal to V^*.

Theorem 4.2.4. *Let $P \in \mathcal{P}(\mathbb{H}^n)$. The measure $\det(\frac{\partial^2 h_P}{\partial q_i \partial \bar{q}_j}) d\, vol$ is supported on the union of $F^{\vee}$ where F runs over all n-dimensional faces of P. For an n-dimensional face F of P the restriction of $\det(\frac{\partial^2 h_P}{\partial q_i \partial \bar{q}_j}) d\, vol$ to $F^{\vee}$ is equal to $\sqrt{vol_{4n}(Q_{F^{\vee}} + Q_{F^{\vee}} \cdot i + Q_{F^{\vee}} \cdot j + Q_{F^{\vee}} \cdot k)} \cdot vol_{F^{\vee}}$ where $Q_{F^{\vee}}$ denotes the unit cube in the span of $F^{\vee}$, and $vol_{F^{\vee}}$ denotes the volume form in the span of $F^{\vee}$ induced by the Euclidean metric on $\mathbb{H}^n$.*

Proof. Let us denote for brevity by μ_P the measure $\det(\frac{\partial^2 h_P}{\partial q_i \partial \bar{q}_j}) d\, vol$. Since the supporting functional h_P is homogeneous of degree 1 then the measure μ_P is homogeneous of degree $3n$, namely for any compact set A and any $\lambda > 0$ one has $\mu_P(\lambda A) = \lambda^{3n} \mu_P(A)$. Next we will need the following claim.

Claim 4.2.5. *Let F be a face of P. Assume that F contains* 0. *Then there exists an open subset $U \subset V^*$ containing $int F^{\vee}$ such that $h_P|_U$ is invariant under translations with respect to span $F^{\vee}$.*

To prove this claim it is enough to observe that one can choose U to be the interior of $\cup_G G^{\vee}$ where G runs over all vertices of F.

Let us now fix a face F of P. By Claim 4.2.5 the restriction of μ_P to $int F^{\vee}$ is a translation invariant measure on $int F^{\vee}$. Hence it is proportional to the Lebesgue measure. We have previously mentioned that μ_P is homogeneous of degree $3n$. Hence if $\dim F \neq n$ the restriction of μ_P to $int F^{\vee}$ vanishes. Let us now assume that $\dim F = n$. Thus the restriction of μ_P to $F^{\vee}$ is proportional to the Lebesgue measure vol_{3n} induced by the Euclidean metric.

Let us denote by U the interior of the union $\cup_G G^{\vee}$ where G runs over all vertices of F. Then U is a non-empty open subset of V^*, and for any $y \in U$ one has

$$h_P(y) = \sup\{y(x) \mid x \in F\} = h_F(y).$$

Hence we may replace P by F, namely we will assume that $\dim P = n$, $F = P$. Let us prove that if $\dim P = n$ then the measure μ_P restricted to $P^\perp$ is equal to $\sqrt{vol(Q_{P^\perp} + Q_{P^\perp}\cdot I + Q_{P^\perp}\cdot J + Q_{P^\perp}\cdot K)}vol_{P^\perp}$. Let us fix an orthonormal basis $\{\xi_1,\dots,\xi_n\}$ in $span P$. We can choose a polytope $\tilde{P} \subset \mathbb{R}^n \subset \mathbb{H}^n$ and an $\mathbb{H}$-linear operator

$$A\colon \mathbb{H}^n \to \mathbb{H}^n$$

such that $A(\tilde{P}) = P$ and for the standard basis $e_1,\dots,e_n$ of $\mathbb{H}^n$

$$A(e_i) = \xi_i,\ i = 1,\dots,n.$$

Then clearly $h_P = h_{\tilde{P}} \circ A^*$. Hence by Proposition 1.3.2 for any $q \in \mathbb{H}^{n*}$ one has

$$\det\left(\frac{\partial^2 h_P}{\partial q_i \partial \bar{q}_j}(q)\right) = \det(AA^*)\det\left(\frac{\partial^2 h_{\tilde{P}}}{\partial q_i \partial \bar{q}_j}(A^* q)\right).$$

We may assume that A is invertible. Let $w = A^* q$. Then we obtain

$$dw = |\det(^{\mathbb{R}}A)|dq = (\det AA^*)^2 dq$$

where the last equality follows from Proposition 1.2.15. Hence we conclude

$$\det\left(\frac{\partial^2 h_P}{\partial q_i \partial \bar{q}_j}(q)\right)dq = \det(AA^*)^{-1}\det\left(\frac{\partial^2 h_{\tilde{P}}}{\partial q_i \partial \bar{q}_j}(w)\right)dw.$$

Lemma 4.2.6. *The measure* $\det(\frac{\partial^2 h_{\tilde{P}}}{\partial q_i \partial \bar{q}_j}(w))dw$ *is equal to the Lebesgue measure on* $\mathbb{R}^{n\perp}$ *induced by the standard Euclidean metric on* $\mathbb{H}^n$ *times* $vol(\tilde{P})$.

Lemma 4.2.6 easily follows from the observation that $\frac{\partial^2 h_{\tilde{P}}}{\partial q_i \partial \bar{q}_j} = \frac{\partial^2 h_{\tilde{P}}}{\partial x_i \partial x_j}$ where $(x_1,\dots,x_n)$ are standard coordinates on $\mathbb{R}^n$, and the fact that if h is the restriction of $h_{\tilde{P}}$ to $\mathbb{R}^{n*}$ then

$$\det\left(\frac{\partial^2 h}{\partial x_i \partial x_j}(w)\right)dw = vol(P)\delta_0,$$

where δ_0 is the delta-measure at the origin 0.

Thus we have to compute $(A^*)^{-1}_*(vol_{\mathbb{R}^{n\perp}})$ as a measure on $(span P)^\perp$. For any two Euclidean spaces $L^{(1)}$, $L^{(2)}$ of the same dimension and for any linear map $\phi\colon V^{(1)} \to$

$V^{(2)}$ let us denote by $|\det\phi|$ the coefficient of the volume distortion, namely for any compact set X of positive measure $|\det\phi| = \frac{vol\,\phi(X)}{vol\,X}$. One can easily prove the following linear algebraic lemma.

Lemma 4.2.7. *Let $V^{(1)}$, $V^{(2)}$ be Euclidean vector spaces of the same dimension. Let*

$$\phi: V^{(1)} \to V^{(2)}$$

be a linear operator. Let $L^{(1)} \subset V^{(1)}$ be a linear subspace. Denote $L^{(2)} = \phi(L^{(1)})$. Assume that $\phi|_{L^{(1)}}: L^{(1)} \to L^{(2)}$ is a linear isomorphism. Set

$$\psi := \phi^*|_{L^{(2)\perp}}: L^{(2)\perp} \to L^{(1)\perp}.$$

Then $|\det\psi| = \frac{|\det\phi|}{|\det\phi|_{L^{(1)}}|}$.

In our situation $L^{(1)} = \mathbb{R}^n$, $\phi|_{L^{(1)}}$ is an isometry. Hence by Lemma 4.2.7 one has

$$(A^*)^{-1}_*(vol_{\mathbb{R}^{n\perp}}) = |\det({}^{\mathbb{R}}A)| \cdot vol_{P^\perp} = (\det(AA^*))^2 vol_{P^\perp}.$$

Hence

$$\det\left(\frac{\partial^2 h_P}{\partial q_i \partial \bar{q}_j}\right)dq = \det(AA^*)vol_{P^\perp} = \sqrt{\det({}^{\mathbb{R}}A)}vol_{P^\perp}.$$

But $\det({}^{\mathbb{R}}A) = vol_{4n}(Q_{\mathrm{span}\,P} + Q_{\mathrm{span}\,P}\cdot I + Q_{\mathrm{span}\,P}\cdot J + Q_{\mathrm{span}\,P}\cdot K)$. This proves Theorem 4.2.4. $\square$

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